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Asymptotic behavior of solutions for a Lotka–Volterra mutualism reaction–diffusion system with time delays☆

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ABSTRACT

This paper is to investigate the asymptotic behavior of solutions for a time-delayed Lotka–Volterra N-species mutualism reaction–diffusion system with homogeneous Neumann boundary condition. It is shown, under a simple condition on the reaction rates, that the system has a unique bounded time-dependent solution and a unique constant positive steady-state solution, and for any nontrivial nonnegative initial function the corresponding time-dependent solution converges to the constant positive steady-state solution as time tends to infinity. This convergence result implies that the trivial steady-state solution and all forms of semitrivial steady-state solutions are unstable, and moreover, the system has no nonconstant positive steady-state solution. A condition ensuring the convergence of the time-dependent solution to one of nonnegative semitrivial steady-state solutions is also given.

1. Introduction

In the field of population dynamics, Lotka–Volterra reaction–diffusion system is an extensively studied class of systems. For the study of such systems, a most important concern is the asymptotic behavior of the time-dependent solution in relation to the steady-state solutions as time tends to infinity. Of particular interest is the determination of the exact limit of the time-dependent solution since many such systems possess multiple steady-state solutions. In this paper, we investigate the asymptotic behavior problem for a Lotka–Volterra N-species mutualism reaction–diffusion system with time delays. The system is given in the form

$$\begin{cases} \partial u_i/\partial t - L_i u_i = u_i \left(a_i - b_i u_i + \sum\limits_{j=1, j\neq i}^{N} c_{ij} u_j + \sum\limits_{j=1, j\neq i}^{N} c'_{ij} (u_j)_{\tau_j} \right) \quad (x \in \Omega,\, t > 0), \\ \partial u_i/\partial \nu = 0 \quad (x \in \partial\Omega,\, t > 0), \\ u_i(x,t) = \eta_i(x,t) \quad (x \in \Omega,\, t \in I_i),\ i = 1, 2, \ldots, N, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. For each $i = 1, 2, \ldots, N$, $(u_i)_{\tau_i} = u_i(x, t - \tau_i)$ with time delay $\tau_i > 0$, $I_i = [-\tau_i, 0]$, and L_i is a diffusion–convection operator given by

$$L_i u_i = D_i(x) \Delta u_i + \mathbf{b}_i(x) \cdot \nabla u_i \tag{1.2}$$

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where Δ and ∇ are the Laplace and gradient operators, and $D_i(x)$ and $\mathbf{b}_i(x)$ are the diffusion and convection coefficients. It is assumed that for each $i = 1, 2, \ldots, N$, $D_i(x)$, $\mathbf{b}_i(x)$ and $\eta_i(x, t)$ are C^α-functions in their respective domains with $D_i(x) > 0$ on $\overline{\Omega} = \Omega \cup \partial\Omega$, $\eta_i(x, t) \geq 0$ on $\Omega \times I_i$, and $\partial\eta_i/\partial\nu = 0$ at $t = 0$. It is also assumed that for each $i, j = 1, 2, \ldots, N$, a_i, b_i, c_{ij} and c'_{ij} are some constants where b_i is positive, c_{ij} and c'_{ij} are nonnegative with $c_{ij} + c'_{ij} > 0$, whereas a_i may be positive or nonpositive. In this system we allow $L_i = 0$ (without the corresponding boundary condition) for some or all i. This means that problem (1.1) may be a coupled system of parabolic and ordinary differential equations.

Since c_{ij} and c'_{ij} are nonnegative with $c_{ij}+c'_{ij} > 0$, system (1.1) is usually called mutualistic or cooperative. In the ecological sense, it describes a cooperative interaction of N-species that benefits each other in a spatial Ω. The unknown u_i represents the population density of the ith species, a_i is the natural growth rate, b_i denotes the respective intraspecific competition, and c_{ij} and c'_{ij} are the interspecific cooperations among species. The boundary condition in (1.1) implies that the population densities do not move across the boundary $\partial\Omega$.

The investigation of the Lotka–Volterra mutualism models in the framework of ordinary differential systems with or without time delays is extensive, and various aspects of the problem, such as the existence of positive solutions and global asymptotic stability of positive periodic solutions, have been discussed (see [1–5] and the references therein). There is also extensive investigation for the present problem (1.1) and various similar problems (see [6–13]), but most of the investigations are devoted to the ones with $N = 2$ and without time delays. In particular, the works in [14–17] are concerned with the coexistence and permanence problems for certain two-species mutualistic models without time delays under Dirichlet boundary condition, and those in [6,7,11,13,18] are for the asymptotic behavior of the solution with Neumann or Robin boundary condition, including the finite time blow-up property. The similar coexistence and asymptotic behavior problems were treated in [19] for a two-species mutualism model with a saturating interaction. Some other studies for two-species mutualism models without time delays can be found in [8,20], where the existence of traveling wave front solutions and time-periodic solutions were discussed. Of recent papers for three-species models, let us mention [9] and [10]. These papers deal with a three-species mutualism model without time delays, and the main concerns there are the blow-up property and the blow-up estimate.

The earlier discussions for the asymptotic behavior problem of reaction–diffusion systems were mostly devoted to coupled systems of two equations with Dirichlet or Robin boundary condition (see [18,21–23]). In recent years, attention has been given to reaction–diffusion systems with three or N population species and with Neumann boundary condition (see [24–28]). In [25], Pao investigated the asymptotic behavior of the time-dependent solution for three 3-species Lotka–Volterra reaction–diffusion systems (two prey–predator systems and one food-chain system), and obtained some simple conditions to ensure the asymptotic convergence of the time-dependent solution to a unique positive steady-state solution for each of the three systems. The same asymptotic convergence problem was discussed by Pao in [26] for a reaction–diffusion system of N-competing species and in [27] for a 3-species competitor–competitor-mutualist reaction–diffusion system. In all these works, possible time delays are taken into consideration in the reaction mechanism and the boundary condition is of Neumann type. Motivated by the above works of Pao, we investigate in this paper the asymptotic behavior of the time-dependent solution for the general N-species mutualism system (1.1). Compared with the solutions of the systems considered in the works of Pao, there is a quite different behavior of solutions in the system (1.1). The solution of (1.1) may blow up in a finite time due to the quasimonotone nondecreasing property of the reaction functions in (1.1) (see [9,10]). Here, we are mainly interested in determining when a bounded solution of (1.1) exists and when it converges to a steady-state solution as time tends to infinity.

Specifically, we show that if the reaction rates b_i, c_{ij} and c'_{ij} form an M-matrix (see (2.1)) then system (1.1) has a unique bounded solution $(u_1, u_2, \ldots, u_N)$, and under an additional simple condition on the reaction rates (see (2.3)), system (1.1) has a unique constant positive steady-state solution $(c_1^*, c_2^*, \ldots, c_N^*)$ and for any nontrivial nonnegative initial function $(\eta_1, \eta_2, \ldots, \eta_N)$ the corresponding time-dependent solution $(u_1, u_2, \ldots, u_N)$ converges to $(c_1^*, c_2^*, \ldots, c_N^*)$ as time tends to infinity. This convergence result implies that the trivial steady-state solution and all forms of semitrivial steady-state solutions are unstable, and moreover, the system (1.1) has no nonconstant positive steady-state solution. In addition, we also give a condition (see (2.7)) so that the time-dependent solution $(u_1, u_2, \ldots, u_N)$ of (1.1) converges to one of nonnegative semitrivial steady-state solutions. In terms of ecological dynamics, the above convergence property gives some coexistence, permanence and extinction results for system (1.1). To obtain these conclusions we use the method of upper and lower solutions which has been widely applied to both continuous and discrete systems (cf. [18,29–32]). Since our conditions depend only on the reaction rates, the same conclusions are directly applicable to the system (1.1) without time delays and to the corresponding ordinary differential system with or without time delays.

The outline of the paper is as follows. In Section 2, we state our main results. Proofs of the results are given in Section 3.

2. The main results

For the sake of presentation, we define some matrices and vectors in terms of the reaction rates. Let $A = (a_{ij})$ be an $N \times N$ matrix defined by the reaction rates b_i, c_{ij} and c'_{ij} (but not by a_i) in the form

$$a_{ii} = b_i, \qquad a_{ij} = -(c_{ij} + c'_{ij}), \quad i, j = 1, 2, \ldots, N, \tag{2.1}$$

and let $\mathbf{r} = (a_1, a_2, \ldots, a_N)^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes the transpose of a row vector. For each $k = 1, 2, \ldots, N$, we denote by A_k the $k \times k$ leading principal submatrix of A and define the vector $\mathbf{r}_k = (a_1, a_2, \ldots, a_k)^{\mathrm{T}}$. In particular, we have $A_N = A$ and

$\mathbf{r}_N = \mathbf{r}$. For the convenience of discussion we make no distinction between row and column vectors. We recall that A is said to be an M-matrix if the inverse A^{-1} exists and is nonnegative (see [33,34]).

Our first result is for the existence of a bounded solution of (1.1).

Theorem 2.1. *Assume that the matrix A is an M-matrix. Then, for any given nonnegative initial function $(\eta_1, \eta_2, \dots, \eta_N)$, system (1.1) has a unique global solution $(u_1, u_2, \dots, u_N)$ and there exist positive constants M_i^* such that*

$$(0, 0, \dots, 0) \le (u_1(x, t), u_2(x, t), \dots, u_N(x, t)) \le (M_1^*, M_2^*, \dots, M_N^*) \quad (x \in \overline{\Omega}, t > 0). \tag{2.2}$$

Moreover, the solution $(u_1(x, t), u_2(x, t), \dots, u_N(x, t))$ is positive for all $x \in \overline{\Omega}$, $t > 0$ if $\eta_i(x, 0) \not\equiv 0$ in Ω for $i = 1, 2, \dots, N$.

To give our results on the asymptotic behavior of the solution we assume that the initial function $(\eta_1, \eta_2, \dots, \eta_N)$ is nontrivial nonnegative in the sense that $\eta_i(x, 0) \not\equiv 0$ in Ω for $i = 1, 2, \dots, N$.

We first consider the case where $a_i > 0$ for some i, say, $i = 1, 2, \dots, N_0$, and $a_i \le 0$ for $i = N_0 + 1, \dots, N$.

Theorem 2.2. *Assume that A is an M-matrix. If, in addition,*

$$a_i + \sum_{j=1}^{N_0} (c_{ij} + c'_{ij}) a_j / b_j > 0, \quad i = N_0 + 1, \dots, N, \tag{2.3}$$

then

(i) *system (1.1) has a unique positive steady-state solution $\mathbf{c}^* = (c_1^*, c_2^*, \dots, c_N^*)$ governed by the algebraic system*

$$A\mathbf{c}^* = \mathbf{r}; \tag{2.4}$$

(ii) *for any nontrivial nonnegative initial function $(\eta_1, \eta_2, \dots, \eta_N)$ the corresponding solution $(u_1(x, t), u_2(x, t), \dots, u_N(x, t))$ of (1.1) converges uniformly to $\mathbf{c}^* = (c_1^*, c_2^*, \dots, c_N^*)$ as $t \to \infty$.*

Remark 2.1. We now give some comments on our conditions in the above theorems. (a) A lot of sufficient (and necessary) conditions ensuring that A is an M-matrix can be found in [33,34]. In particular, a simple and easily verified sufficient condition is that for a certain i,

$$\sum_{j=1}^{N} (c_{ij} + c'_{ij}) < b_i. \tag{2.5}$$

(b) In fact, the condition (2.3) is trivially satisfied if $a_i = 0$ for all $i = N_0 + 1, \dots, N$. Hence it is needed only for the case of $a_i < 0$ for some i.

It is obvious that system (1.1) has the trivial steady-state solution $(0, 0, \dots, 0)$ and various forms of semitrivial steady-state solutions in the sense that each solution has at least one component zero. Since the convergence of the solution $(u_1(x, t), u_2(x, t), \dots, u_N(x, t))$ to $(c_1^*, c_2^*, \dots, c_N^*)$ in Theorem 2.2 is for every nontrivial nonnegative initial function, the trivial steady-state solution and all forms of semitrivial steady-state solutions of (1.1) are unstable (with respect to nontrivial nonnegative initial perturbations). Moreover, the uniqueness of positive steady-state solution $(c_1^*, c_2^*, \dots, c_N^*)$ implies that system (1.1) has no nonconstant positive steady-state solution. We summarize these observations in the following corollary.

Corollary 2.1. *Under the conditions in Theorem 2.2, the trivial steady-state solution and all forms of semitrivial steady-state solutions of (1.1) are unstable, and system (1.1) has no nonconstant positive steady-state solution.*

Theorem 2.2 implies that under condition (2.3), the constant steady-state solution $\mathbf{c}^* = (c_1^*, c_2^*, \dots, c_N^*)$ is a global attractor. In case this condition is not satisfied, then one of the semitrivial steady-state solutions may become a global attractor. To give such a sufficient condition we observe from Theorem 2.2 that if A is an M-matrix then the algebraic system

$$A\mathbf{d}' = \mathbf{r}' \tag{2.6}$$

has a unique positive solution $\mathbf{d}' = (d'_1, d'_2, \dots, d'_N)^{\mathrm{T}}$, where $\mathbf{r}' = (a_1, a_2, \dots, a_{N_0}, 0, \dots, 0)^{\mathrm{T}}$. In view of this positive solution we have the following global attraction property of the semitrivial steady-state solutions.

Theorem 2.3. *Assume that A is an M-matrix. If, in addition,*

$$a_i + \sum_{j=1}^{N_0} (c_{ij} + c'_{ij}) d'_j + \sum_{j=i+1}^{N} (c_{ij} + c'_{ij}) d'_j < 0, \quad i = N_0 + 1, \dots, N, \tag{2.7}$$

where $\mathbf{d}' = (d'_1, d'_2, \dots, d'_N)^{\mathrm{T}}$ is the positive solution of (2.6), then

(i) *system* (1.1) *has a semitrivial nonnegative steady-state solution* $\mathbf{c}^* = (c_1^*, \ldots, c_{N_0}^*, 0, \ldots, 0)$, *where* $\mathbf{c}_0^* = (c_1^*, c_2^*, \ldots, c_{N_0}^*)^{\mathrm{T}}$ *is the unique positive solution of the algebraic system* $A_{N_0}\mathbf{c}_0^* = \mathbf{r}_{N_0}$;

(ii) *for any nontrivial nonnegative initial function* $(\eta_1, \eta_2, \ldots, \eta_N)$ *the corresponding solution* $(u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ *of* (1.1) *converges uniformly to the semitrivial steady-state solution* $\mathbf{c}^* = (c_1^*, c_2^*, \ldots, c_{N_0}^*, 0, 0, \ldots, 0)$ *as* $t \to \infty$.

As a limit situation of Theorem 2.2 we have the following asymptotic behavior result for the case of all $a_i \le 0$.

Theorem 2.4. *Assume that* A *is an* M*-matrix and* $a_i \le 0$ *for all* $i = 1, 2, \ldots, N$. *Then*

(i) *trivial solution* $(0, 0, \ldots, 0)$ *is a unique nonnegative steady-state solution of system* (1.1);

(ii) *for any nontrivial nonnegative initial function* $(\eta_1, \eta_2, \ldots, \eta_N)$ *the corresponding solution* $(u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ *of* (1.1) *converges uniformly to trivial solution* $(0, 0, \ldots, 0)$ *as* $t \to \infty$.

Finally, we consider the special case of two species (i.e., $N = 2$). For this case, the condition (2.7) can be weakened so that there is no gap between the conditions (2.3) and (2.7).

Theorem 2.5. *Assume* $N = 2$ *and* $b_1b_2 > (c_{12} + c'_{12})(c_{21} + c'_{21})$. *We have*

(i) *for any given nonnegative initial function* (η_1, η_2), *system* (1.1) *has a unique bounded nonnegative global solution* (u_1, u_2);

(ii) *if either*

$$a_1 > 0, \quad a_2b_1 + a_1(c_{21} + c'_{21}) > 0 \quad \text{or} \quad a_2 > 0, \quad a_1b_2 + a_2(c_{12} + c'_{12}) > 0, \tag{2.8}$$

then for any nontrivial nonnegative initial function (η_1, η_2) *the corresponding solution* $(u_1(x,t), u_2(x,t))$ *of* (1.1) *converges uniformly to* (c_1^*, c_2^*) *as* $t \to \infty$, *where* (c_1^*, c_2^*) *is a unique positive steady-state solution of* (1.1) *and is given by*

$$c_1^* = \frac{a_1b_2 + a_2(c_{12} + c'_{12})}{b_1b_2 - (c_{12} + c'_{12})(c_{21} + c'_{21})}, \qquad c_2^* = \frac{a_2b_1 + a_1(c_{21} + c'_{21})}{b_1b_2 - (c_{12} + c'_{12})(c_{21} + c'_{21})}; \tag{2.9}$$

(iii) *if*

$$a_1 > 0, \quad a_2b_1 + a_1(c_{21} + c'_{21}) \le 0, \tag{2.10}$$

then for any nontrivial nonnegative initial function (η_1, η_2) *the corresponding solution* $(u_1(x,t), u_2(x,t))$ *of* (1.1) *converges uniformly to* $(a_1/b_1, 0)$ *as* $t \to \infty$;

(iv) *if*

$$a_2 > 0, \quad a_1b_2 + a_2(c_{12} + c'_{12}) \le 0, \tag{2.11}$$

then for any nontrivial nonnegative initial function (η_1, η_2) *the corresponding solution* $(u_1(x,t), u_2(x,t))$ *of* (1.1) *converges uniformly to* $(0, a_2/b_2)$ *as* $t \to \infty$;

(v) *if* $a_1 \le 0$ *and* $a_2 \le 0$ *then for any nontrivial nonnegative initial function* (η_1, η_2) *the corresponding solution* $(u_1(x,t), u_2(x,t))$ *of* (1.1) *converges uniformly to* $(0, 0)$ *as* $t \to \infty$, *and the trivial solution* $(0, 0)$ *is a unique nonnegative steady-state solution of* (1.1).

Remark 2.2. The results in Theorem 2.5 for the case without time delays have also been obtained in [13]. In the case with $a_1 > 0, a_2 > 0$ and without time delays, Theorem 2.5 is similar to Theorems 6.1–6.5 in Section 12.6 of [18], where very general boundary conditions are considered.

Remark 2.3. (a) In the above theorems we have assumed that $\eta_i(x, 0) \not\equiv 0$ in Ω for all $i = 1, 2, \ldots, N$. If $\eta_i(x, 0) \equiv 0$ for some i (say, $i = 1, 2, \ldots, N_1$) and $\eta_i(x, 0) \not\equiv 0$ for $i = N_1 + 1, \ldots, N$ then $u_i(x,t) \equiv 0$ in $\Omega \times [0, \infty)$ for $i = 1, 2, \ldots, N_1$ and $u_i(x,t) > 0$ in $\Omega \times (0, \infty)$ for $i = N_1 + 1, \ldots, N$. In this situation, system (1.1) is reduced to an $(N - N_1)$-subsystem and all the conclusions in the above theorems are applicable to the $(N - N_1)$-subsystem. (b) Since the conditions in the above theorems are independent of time delays and the effect of diffusion–convection, all the above conclusions hold true for the system (1.1) without time delays and for its corresponding ordinary differential system with or without time delays.

3. Proofs of the main results

To prove the main results in Section 2 we use the method of upper and lower solutions developed in [24–27] and some comparisons of scalar reaction–diffusion equations.

Given any domain D we denote by $C(D)$ the set of continuous functions in D, and by $C^{2,1}(D)$ the set of functions that are twice continuously differentiable in x and once continuously differentiable in t for all $(x,t) \in D$. For each $i = 1, 2, \ldots, N$, we let $\overline{Q}_i = \overline{\Omega} \times [-\tau_i, +\infty)$ and $Q_i = \Omega \times [-\tau_i, +\infty)$. Define the product spaces $\mathcal{Q} = \prod_{i=1}^N C(\overline{Q}_i)$ and $\mathcal{Q}^{2,1} = \prod_{i=1}^N C^{2,1}(Q_i)$. We call two functions $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_N)$ and $\widehat{\mathbf{u}} = (\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_N)$ in $\mathcal{Q} \cap \mathcal{Q}^{2,1}$ a pair of ordered upper and lower solutions of (1.1) if they satisfy $\widetilde{\mathbf{u}} \ge \widehat{\mathbf{u}}$ and the relations in (1.1) where the equality sign $=$ is replaced by the inequality signs $\ge$ and $\le$, respectively. Here and in what follows, the inequality between vectors is to be understood componentwise.

For convenience, we denote the reaction functions in (1.1) by $f_i(\mathbf{u}, \mathbf{v})$, i.e.,

$$f_i(\mathbf{u}, \mathbf{v}) = u_i \left(a_i - b_i u_i + \sum_{j=1, j\neq i}^{N} c_{ij} u_j + \sum_{j=1, j\neq i}^{N} c'_{ij} v_j \right), \quad i = 1, 2, \ldots, N, \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_N)$. Let $\widetilde{\mathbf{c}} = (\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_N)$ and $\widehat{\mathbf{c}} = (\widehat{c}_1, \widehat{c}_2, \ldots, \widehat{c}_N)$ be a pair of constant vectors such that $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}} \geq \mathbf{0}$ and

$$f_i(\widetilde{\mathbf{c}}, \widetilde{\mathbf{c}}) \leq 0 \leq f_i(\widehat{\mathbf{c}}, \widehat{\mathbf{c}}), \quad i = 1, 2, \ldots, N. \tag{3.2}$$

Then the pair $\widetilde{\mathbf{c}}$ and $\widehat{\mathbf{c}}$ form a pair of ordered upper and lower solutions of (1.1) if and only if $\widehat{c}_i \leq \eta_i(x, t) \leq \widetilde{c}_i$ for $x \in \Omega$ and $t \in I_i$ $(i = 1, 2, \ldots, N)$. Define the sector $\mathcal{S}$ by

$$\mathcal{S} = \{\mathbf{c} \in \mathbf{R}^N; \widehat{\mathbf{c}} \leq \mathbf{c} \leq \widetilde{\mathbf{c}}\}. \tag{3.3}$$

Let K_i be any positive constant satisfying

$$K_i \geq \max\left\{ -\frac{\partial f_i}{\partial u_i}(\mathbf{u}, \mathbf{v}); \mathbf{u}, \mathbf{v} \in \mathcal{S} \right\}, \quad i = 1, 2, \ldots, N. \tag{3.4}$$

Starting from $\widetilde{\mathbf{c}}$ or $\widehat{\mathbf{c}}$ we construct a sequence of constant vectors $\{\mathbf{c}^{(m)}\} = \{(c_1^{(m)}, c_2^{(m)}, \ldots, c_N^{(m)})\}$ from the following recursion relation

$$c_i^{(m)} = c_i^{(m-1)} + f_i(\mathbf{c}^{(m-1)}, \mathbf{c}^{(m-1)})/K_i, \quad i = 1, 2, \ldots, N. \tag{3.5}$$

Denote $\{\mathbf{c}^{(m)}\}$ by $\{\overline{\mathbf{c}}^{(m)}\}$ if $\mathbf{c}^{(0)} = \widetilde{\mathbf{c}}$ and by $\{\underline{\mathbf{c}}^{(m)}\}$ if $\mathbf{c}^{(0)} = \widehat{\mathbf{c}}$.

We now review some known results from [19,24] for the present problem (1.1).

Theorem 3.1. *Let $\widetilde{\mathbf{c}}$ and $\widehat{\mathbf{c}}$ be a pair of constant vectors satisfying $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}} \geq \mathbf{0}$ and the relation (3.2). We have the following results.*

(i) *If $\widetilde{\mathbf{c}}$ and $\widehat{\mathbf{c}}$ are a pair of ordered upper and lower solutions of (1.1) then system (1.1) has a unique solution $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ in $\mathcal{S}$.*

(ii) *The sequences $\{\overline{\mathbf{c}}^{(m)}\}$ and $\{\underline{\mathbf{c}}^{(m)}\}$ from (3.5) converge to respective limits $\overline{\mathbf{c}}$ and $\underline{\mathbf{c}}$ that satisfy*

$$\widehat{\mathbf{c}} \leq \underline{\mathbf{c}} \leq \overline{\mathbf{c}} \leq \widetilde{\mathbf{c}}, \quad f_i(\overline{\mathbf{c}}, \overline{\mathbf{c}}) = 0, \quad f_i(\underline{\mathbf{c}}, \underline{\mathbf{c}}) = 0, \quad i = 1, 2, \ldots, N. \tag{3.6}$$

(iii) *If $\overline{\mathbf{c}} = \underline{\mathbf{c}} (\equiv \mathbf{c}^*)$ and there exists a finite $t^* \geq 0$ such that*

$$\widehat{\mathbf{c}} \leq \mathbf{u}(x, t^*) \leq \widetilde{\mathbf{c}}, \quad x \in \Omega, \tag{3.7}$$

then for any nonnegative initial function $(\eta_1, \eta_2, \ldots, \eta_N)$ the corresponding solution $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ of (1.1) converges uniformly to $\mathbf{c}^*$ as $t \to \infty$.

Theorem 3.2. *Let $u(x, t)$ be a function in $C(\overline{\Omega} \times [t_0, \infty)) \cap C^{2,1}(\Omega \times (t_0, \infty))$ such that*

$$\begin{cases} \partial u/\partial t - L_i u \geq au(b - u) & (x \in \Omega, t > t_0), \\ \partial u/\partial \nu \geq 0 & (x \in \partial\Omega, t > t_0), \\ u(x, t_0) \geq 0, \not\equiv 0 & (x \in \Omega), \end{cases} \tag{3.8}$$

where a and b are two positive constants. Then for arbitrary positive constant ε, there exists a finite $t^* > t_0$ such that

$$u(x, t) \geq b - \varepsilon \quad (x \in \overline{\Omega}, t \geq t^*). \tag{3.9}$$

Theorem 3.3. *Let $u(x, t)$ be a function in $C(\overline{\Omega} \times [t_0, \infty)) \cap C^{2,1}(\Omega \times (t_0, \infty))$ such that*

$$\begin{cases} \partial u/\partial t - L_i u \leq -au & (x \in \Omega, t > t_0), \\ \partial u/\partial \nu \leq 0 & (x \in \partial\Omega, t > t_0), \\ u(x, t_0) \geq 0, \not\equiv 0 & (x \in \Omega), \end{cases} \tag{3.10}$$

where the constant a is positive. Then for arbitrary positive constant ε, there exists a finite $t^* > t_0$ such that

$$u(x, t) \leq \varepsilon \quad (x \in \overline{\Omega}, t \geq t^*). \tag{3.11}$$

We now prove the main results in Section 2 using the above theorems.

Proof of Theorem 2.1. We know from the matrix theory that the matrix A is an M-matrix if and only if there exists a positive vector $\mathbf{M} = (M_1, M_2, \ldots, M_N)^{\mathrm{T}}$ such that $A\mathbf{M} > \mathbf{0}$ (see [33,34]). Choose a sufficiently large positive constant δ such that

$\delta A\mathbf{M} \geq \mathbf{r}$ and $\delta M_i \geq \eta_i(x,t)$ for $x \in \Omega$ and $t \in I_i$ $(i = 1, 2, \ldots, N)$. Then the pair $\widetilde{\mathbf{c}} = \delta\mathbf{M}$ and $\widehat{\mathbf{c}} = \mathbf{0}$ satisfy relation (3.2), and moreover, they are ordered upper and lower solutions of (1.1). Since δ can be chosen arbitrarily large, we conclude from Theorem 3.1(i) that there exist positive constants M_i^* such that system (1.1) has a unique nonnegative global solution $(u_1, u_2, \ldots, u_N)$ which satisfies relation (2.2). When $\eta_i(x, 0) \not\equiv 0$ in Ω for $i = 1, 2, \ldots, N$, the positivity of $(u_1, u_2, \ldots, u_N)$ follows from the maximum principle. $\square$

Proof of Theorem 2.2. We divide the proof into four parts as follows.

Part I: *Finding a pair of constant vectors* $\widetilde{\mathbf{c}}$ *and* $\widehat{\mathbf{c}}$ *satisfying* $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}} \geq \mathbf{0}$ *and* (3.2). For convenience, we let

$$e_i = \sum_{j=1}^{N_0} (c_{ij} + c'_{ij}) a_j/b_j, \qquad e'_i = \sum_{j=1}^{N_0} (c_{ij} + c'_{ij}), \quad i = N_0 + 1, \ldots, N. \tag{3.12}$$

Then the condition (2.3) implies that $a_i + e_i > 0$ for $i = N_0 + 1, \ldots, N$. For a given positive constant ε satisfying $\varepsilon < a_i/b_i$ for $i = 1, 2, \ldots, N_0$ and $\varepsilon < (a_i + e_i)/(e'_i + b_i)$ for $i = N_0 + 1, \ldots, N$, we define the positive vector $\widehat{\mathbf{c}} = (\widehat{c}_1, \widehat{c}_2, \ldots, \widehat{c}_N)$ by $\widehat{c}_i = a_i/b_i - \varepsilon$ for $i = 1, 2, \ldots, N_0$ and $\widehat{c}_i = \left(a_i + e_i - (e'_i + b_i)\varepsilon\right)/b_i$ for $i = N_0 + 1, \ldots, N$. Let $\mathbf{M}$ be the positive vector satisfying $A\mathbf{M} > \mathbf{0}$, and let $\widetilde{\mathbf{c}} = \delta\mathbf{M}$ where δ is sufficiently large such that $A\widetilde{\mathbf{c}} \geq \mathbf{r}$ and $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}}$. A simple calculation shows that $\widetilde{\mathbf{c}}$ and $\widehat{\mathbf{c}}$ satisfy relation (3.2).

Part II: *Showing the limits* $\overline{\mathbf{c}}$ *and* $\underline{\mathbf{c}}$ *of the sequences* $\{\overline{\mathbf{c}}^{(m)}\}$ *and* $\{\underline{\mathbf{c}}^{(m)}\}$ *from* (3.5) *coincide*. By Theorem 3.1(ii), the sequences $\{\overline{\mathbf{c}}^{(m)}\}$ and $\{\underline{\mathbf{c}}^{(m)}\}$ from (3.5) with $\overline{\mathbf{c}}^{(0)} = \widetilde{\mathbf{c}}$ and $\underline{\mathbf{c}}^{(0)} = \widehat{\mathbf{c}}$ converge to the limits $\overline{\mathbf{c}}$ and $\underline{\mathbf{c}}$ that satisfy $\overline{\mathbf{c}} \geq \underline{\mathbf{c}} \geq \widehat{\mathbf{c}} > \mathbf{0}$ and

$$f_i(\overline{\mathbf{c}}, \overline{\mathbf{c}}) = 0, \qquad f_i(\underline{\mathbf{c}}, \underline{\mathbf{c}}) = 0, \quad i = 1, 2, \ldots, N. \tag{3.13}$$

Since $\overline{\mathbf{c}} \geq \underline{\mathbf{c}} > \mathbf{0}$, the above equations, in terms of the matrix A, are equivalent to the algebraic systems $A\overline{\mathbf{c}} = \mathbf{r}$ and $A\underline{\mathbf{c}} = \mathbf{r}$. The nonsingular property of A ensures $\overline{\mathbf{c}} = \underline{\mathbf{c}} (\equiv \mathbf{c}^*)$.

Part III: *Verifying the convergence of the solution* $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ *to* $\mathbf{c}^*$ *as* $t \to \infty$. By Theorem 2.1, for any given nontrivial nonnegative initial function $(\eta_1, \eta_2, \ldots, \eta_N)$, the solution $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ is bounded and nonnegative, and thus we have from (1.1) that for $i = 1, 2, \ldots, N_0$,

$$\begin{cases} \partial u_i/\partial t - L_i u_i \geq u_i(a_i - b_i u_i) \quad (x \in \Omega, t > 0), \\ \partial u_i/\partial \nu = 0 \quad (x \in \partial\Omega, t > 0), \\ u_i(x, 0) \geq 0, \not\equiv 0 \quad (x \in \Omega). \end{cases} \tag{3.14}$$

An application of Theorem 3.2 to the above relation shows that there exists a finite $t_1 > 0$ so that $u_i(x,t) \geq a_i/b_i - \varepsilon$ for all $x \in \overline{\Omega}$, $t \geq t_1$ and every $i = 1, 2, \ldots, N_0$, where ε is the given positive constant in Part I. Using this estimate we have again from (1.1) that for $i = N_0 + 1, \ldots, N$,

$$\begin{cases} \partial u_i/\partial t - L_i u_i \geq u_i(a_i + e_i - e'_i\varepsilon - b_i u_i) \quad (x \in \Omega, t > t_1 + \max\limits_i \tau_i), \\ \partial u_i/\partial \nu = 0 \quad (x \in \partial\Omega, t > t_1 + \max\limits_i \tau_i), \\ u(x, t_1 + \max\limits_i \tau_i) > 0 \quad (x \in \Omega). \end{cases} \tag{3.15}$$

By Theorem 3.2, there exists a finite $t_2 > t_1 + \max_i \tau_i$ such that $u_i(x,t) \geq (a_i + e_i - (e'_i + b_i)\varepsilon)/b_i$ for all $x \in \overline{\Omega}$, $t \geq t_2$ and $i = N_0 + 1, \ldots, N$. Define $t^* = t_2$ and take a sufficiently large δ in the definition of $\widetilde{\mathbf{c}}$. We conclude $\widehat{\mathbf{c}} \leq \mathbf{u}(x, t^*) \leq \widetilde{\mathbf{c}}$ for all $x \in \Omega$. This proves the property (3.7). It follows from Theorem 3.1(iii) that the solution $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ converges uniformly to $\mathbf{c}^*$ as $t \to \infty$.

Part IV: *Proving the uniqueness of the steady-state solution* $\mathbf{c}^*$. Since $\mathbf{c}^*$ is governed by the algebraic system (2.4), it is obvious that $\mathbf{c}^*$ is a constant positive steady-state solution of (1.1). To prove the uniqueness of $\mathbf{c}^*$, we assume that $\mathbf{u}_s(x) = (u_{s,1}(x), u_{s,2}(x), \ldots, u_{s,N}(x))$ is another positive steady-state solution of (1.1). Define $u_i(x,t) = u_{s,i}(x)$ for all $(x,t) \in \overline{Q}_i$ and all $i = 1, 2, \ldots, N$. Then $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ is the solution of (1.1) with the initial function $(\eta_1, \eta_2, \ldots, \eta_N) = (u_{s,1}, u_{s,2}, \ldots, u_{s,N})$. By the convergence result proved in Part III, the solution $\mathbf{u}(x,t)$ converges uniformly to $\mathbf{c}^*$ as $t \to \infty$. This implies $\mathbf{u}_s \equiv \mathbf{c}^*$, and thus the uniqueness of $\mathbf{c}^*$. $\square$

Proof of Theorem 2.3. The proof follows from the similar argument as that in the proof of Theorem 2.2, and we give a sketch. Define

$$\begin{aligned} s_i &= \sum_{j=1, j\neq i}^{N} (c_{ij} + c'_{ij}), \quad i = 1, 2, \ldots, N, \\ s'_i &= \begin{cases} \displaystyle\sum_{j=1, j\neq i}^{N_0} (c_{ij} + c'_{ij}) d'_j, \quad i = 1, 2, \ldots, N_0, \\ \displaystyle\sum_{j=1}^{N_0} (c_{ij} + c'_{ij}) d'_j + \sum_{j=i+1}^{N} (c_{ij} + c'_{ij}) d'_j, \quad i = N_0 + 1, \ldots, N. \end{cases} \end{aligned} \tag{3.16}$$

Then by (2.6)

$$b_i d_i' - a_i - s_i' = \sum_{j=N_0+1}^{N} (c_{ij} + c_{ij}') d_j' > 0, \quad i = 1, 2, \ldots, N_0, \tag{3.17}$$

and by (2.7)

$$-a_i - s_i' > 0, \quad i = N_0 + 1, \ldots, N. \tag{3.18}$$

Let ε be a positive constant such that

$$\varepsilon < \min_{1 \le i \le N_0} \{(b_i d_i' - a_i - s_i')/s_i, a_i/b_i\}, \qquad \varepsilon < \min_{N_0+1 \le i \le N} \{(-a_i - s_i')/s_i\}. \tag{3.19}$$

Define $\widetilde{\mathbf{c}} = (\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_N)$ and $\widehat{\mathbf{c}} = (\widehat{c}_1, \widehat{c}_2, \ldots, \widehat{c}_N)$ by $\widetilde{c}_i = d_i' + \varepsilon, \widehat{c}_i = a_i/b_i - \varepsilon$ for $i = 1, 2, \ldots, N_0$ and $\widetilde{c}_i = \varepsilon, \widehat{c}_i = 0$ for $i = N_0 + 1, \ldots, N$. Then the pair $\widetilde{\mathbf{c}}$ and $\widehat{\mathbf{c}}$ satisfy the relation (3.2) and $\widetilde{\mathbf{c}} \ge \widehat{\mathbf{c}}$ (Notice that $d_i' \ge a_i/b_i$ for all $i = 1, 2, \ldots, N$).

By Theorem 3.1(ii), the sequences $\{\overline{\mathbf{c}}^{(m)}\}$ and $\{\underline{\mathbf{c}}^{(m)}\}$ from (3.5) with $\overline{\mathbf{c}}^{(0)} = \widetilde{\mathbf{c}}$ and $\underline{\mathbf{c}}^{(0)} = \widehat{\mathbf{c}}$ converge to the limits $\overline{\mathbf{c}} = (\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_N)$ and $\underline{\mathbf{c}} = (\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_N)$ satisfying $\widetilde{\mathbf{c}} \ge \overline{\mathbf{c}} \ge \underline{\mathbf{c}} \ge \widehat{\mathbf{c}}$ and relation (3.13). Since $\widehat{c}_i > 0$ for $i = 1, 2, \ldots, N_0$ and $\widehat{c}_i = 0$ for $i = N_0 + 1, \ldots, N$, we have that $\underline{c}_i = 0$ for $i = N_0 + 1, \ldots, N$ and $\underline{\mathbf{c}}_0 = (\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_{N_0})^T$ is governed by $A_{N_0}\underline{\mathbf{c}}_0 = \mathbf{r}_{N_0}$. On the other hand, the limits $\overline{c}_i$ satisfy

$$a_i - b_i \overline{c}_i + \sum_{j=1, j \ne i}^{N} (c_{ij} + c_{ij}') \overline{c}_j = 0, \quad i = 1, 2, \ldots, N_0, \tag{3.20}$$

and

$$\overline{c}_i \left(a_i - b_i \overline{c}_i + \sum_{j=1, j \ne i}^{N} (c_{ij} + c_{ij}') \overline{c}_j \right) = 0, \quad i = N_0 + 1, \ldots, N. \tag{3.21}$$

It follows from $\overline{c}_i \le d_i' + \varepsilon$ for $i = 1, 2, \ldots, N_0$ and $\overline{c}_i \le \varepsilon$ for $i = N_0 + 1, \ldots, N$ that

$$a_i - b_i \overline{c}_i + \sum_{j=1, j \ne i}^{N} (c_{ij} + c_{ij}') \overline{c}_j \le a_i + s_i' + s_i \varepsilon < 0, \quad i = N_0 + 1, \ldots, N. \tag{3.22}$$

By this relation and (3.21), we conclude $\overline{c}_i = 0$ for $i = N_0 + 1, \ldots, N$ and thus by (3.20), $\overline{\mathbf{c}}_0 = (\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{N_0})^T$ is also governed by $A_{N_0}\overline{\mathbf{c}}_0 = \mathbf{r}_{N_0}$. The nonsingular property of A_{N_0} implies $\overline{\mathbf{c}}_0 = \underline{\mathbf{c}}_0$. This proves that the limits $\overline{\mathbf{c}} = \underline{\mathbf{c}} (\equiv \mathbf{c}^* = (c_1^*, c_2^*, \ldots, c_{N_0}^*, 0, 0, \ldots, 0))$ and $\mathbf{c}_0^* = (c_1^*, c_2^*, \ldots, c_{N_0}^*)^T$ is the unique positive solution of $A_{N_0}\mathbf{c}_0^* = \mathbf{r}_{N_0}$. Therefore by Theorem 3.1(iii), the solution $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), \ldots, u_N(x, t))$ of (1.1) converges uniformly to $\mathbf{c}^* = (c_1^*, c_2^*, \ldots, c_{N_0}^*, 0, 0, \ldots, 0)$ as $t \to \infty$, provided for all $x \in \Omega$ and some $t^* > 0$,

$$a_i/b_i - \varepsilon \le u_i(x, t^*) \le d_i' + \varepsilon, \quad i = 1, 2, \ldots, N_0, \tag{3.23}$$

and

$$0 \le u_i(x, t^*) \le \varepsilon, \quad i = N_0 + 1, \ldots, N. \tag{3.24}$$

The first inequality in (3.23) has been proved in the proof of Theorem 2.2 while the first inequality in (3.24) is obvious due to the nonnegative property of the solution. To prove the second inequalities in (3.23) and (3.24), we consider system (1.1) where $a_i > 0$ for $i = 1, 2, \ldots, N_0$ and $a_i = 0$ for $i = N_0 + 1, \ldots, N$. By Theorem 2.1, this system has a unique bounded solution, denoted by $\mathbf{v}(x, t)$, and by Theorem 2.2, it converges uniformly to the positive solution $\mathbf{d}' = (d_1', d_2', \ldots, d_N')^T$ of (2.6) as $t \to \infty$. A comparison between $\mathbf{v}(x, t)$ and $\mathbf{u}(x, t)$ shows that $\mathbf{u}(x, t) \le \mathbf{v}(x, t)$ for $x \in \Omega$ and $t > 0$. This implies that there exists $t_1 > 0$ such that $u_i(x, t) \le d_i' + \varepsilon$ $(i = 1, 2, \ldots, N)$ for $x \in \Omega$ and all $t \ge t_1$. Using this upper bound in (1.1) leads to that for $i = N_0 + 1$,

$$u_i \left(a_i - b_i u_i + \sum_{j=1, j \ne i}^{N} c_{ij} u_j + \sum_{j=1, j \ne i}^{N} c_{ij}' (u_j)_{\tau_j} \right) \le (a_i + s_i' + s_i \varepsilon) u_i \quad (x \in \Omega, t > t_1 + \tau_i).$$

Since $a_{N_0+1} + s_{N_0+1}' + s_{N_0+1}\varepsilon < 0$, an application of Theorem 3.3 to the above relation yields that there exists a finite $t_2 > 0$ such that $u_{N_0+1}(x, t) \le \varepsilon$ for $x \in \Omega$ and all $t \ge t_2$. A similar argument gives $u_i(x, t) \le \varepsilon$ $(i = N_0 + 2, \ldots, N)$ for $x \in \Omega$ and all $t \ge t_2$ (with possibly a different t_2). This proves the second inequalities in (3.23) and (3.24). □

Proof of Theorem 2.4. Let η be an arbitrary positive constant. Since $a_i \leq 0$, we have from a comparison that $\mathbf{u}(x,t) \leq \mathbf{u}_\eta(x,t)$, where $\mathbf{u}_\eta(x,t)$ denotes the solution of (1.1) with $a_1 = \eta$ and $a_i = 0$ for $i = 2, 3, \ldots, N$. By Theorem 2.2, $\mathbf{u}_\eta(x,t)$ converges uniformly to the positive solution $\mathbf{c}^*_\eta$ of (2.4) where $\mathbf{r} = (\eta, 0, \ldots, 0)^{\mathrm{T}}$, as $t \to \infty$. Consequently, for any given positive constant ε, there exists a finite t^* such that $\mathbf{u}(x,t) \leq \mathbf{c}^*_\eta + \frac{1}{2}\boldsymbol{\varepsilon}$ for $x \in \Omega$ and $t \geq t^*$, where $\boldsymbol{\varepsilon} = (\varepsilon, \varepsilon, \ldots, \varepsilon)^{\mathrm{T}}$. It is easily shown that $\mathbf{c}^*_\eta$ converges to $\mathbf{0} = (0, 0, \ldots, 0)^{\mathrm{T}}$ as $\eta \to 0$. This implies that $\mathbf{c}^*_\eta \leq \frac{1}{2}\boldsymbol{\varepsilon}$ for some small η, and thus $\mathbf{0} \leq \mathbf{u}(x,t) \leq \boldsymbol{\varepsilon}$ for $x \in \Omega$ and $t \geq t^*$. This proves that the solution $\mathbf{u}(x,t)$ converges uniformly to trivial solution $(0, 0, \ldots, 0)$ as $t \to \infty$.

Assume that $\mathbf{u}_s(x) = (u_{s,1}(x), u_{s,2}(x), \ldots, u_{s,N}(x))$ is a nonnegative steady-state solution of (1.1). Let $u_i(x,t) = u_{s,i}(x)$ for $(x,t) \in \overline{Q}_i$ and all $i = 1, 2, \ldots, N$. Then $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ is the solution of (1.1) with the initial function $\eta_i(x,t) = u_{s,i}(x)$ $(i = 1, 2, \ldots, N)$. By the convergence result proved above and Remark 2.3(a), $\mathbf{u}(x,t)$ converges uniformly to $(0, 0, \ldots, 0)$ as $t \to \infty$. This implies $\mathbf{u}_s(x) \equiv (0, 0, \ldots, 0)$, and thus the uniqueness of trivial solution. □

Proof of Theorem 2.5. For the case $N = 2$, A is an M-matrix if and only if $b_1 b_2 > (c_{12} + c'_{12})(c_{21} + c'_{21})$ (see [33]). The conclusions in (i), (ii) and (v) follow directly from Theorems 2.1, 2.2 and 2.4, respectively.

We now prove the conclusion in (iii). For any positive constant $\varepsilon < a_1/b_1$, we define $\widehat{\mathbf{c}} = (\widehat{c}_1, \widehat{c}_2)$ by $\widehat{c}_1 = a_1/b_1 - \varepsilon$ and $\widehat{c}_2 = 0$. Let $\mathbf{M}$ be the positive vector satisfying $A\mathbf{M} > \mathbf{0}$, and define $\widetilde{\mathbf{c}} = \delta\mathbf{M}$ where δ is sufficiently large such that $A\widetilde{\mathbf{c}} \geq \mathbf{r}$ and $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}}$. It is easily verified that $\widetilde{\mathbf{c}}$ and $\widehat{\mathbf{c}}$ satisfy relation (3.2). Therefore by Theorem 3.1(ii), the sequences $\{\overline{\mathbf{c}}^{(m)}\}$ and $\{\underline{\mathbf{c}}^{(m)}\}$ from (3.5) with $\overline{\mathbf{c}}^{(0)} = \widetilde{\mathbf{c}}$ and $\underline{\mathbf{c}}^{(0)} = \widehat{\mathbf{c}}$ converge to the limits $\overline{\mathbf{c}} = (\overline{c}_1, \overline{c}_2)$ and $\underline{\mathbf{c}} = (\underline{c}_1, \underline{c}_2)$ which satisfy $\widetilde{\mathbf{c}} \geq \overline{\mathbf{c}} \geq \underline{\mathbf{c}} \geq \widehat{\mathbf{c}}$ and relation (3.13) (with $N = 2$). Since $\widehat{c}_1 = a_1/b_1 - \varepsilon > 0$ and $\widehat{c}_2 = 0$, the relation (3.13) is reduced to

$$\begin{aligned} &a_1 - b_1\overline{c}_1 + (c_{12} + c'_{12})\overline{c}_2 = 0, \qquad \overline{c}_2(a_2 - b_2\overline{c}_2 + (c_{21} + c'_{21})\overline{c}_1) = 0, \\ &a_1 - b_1\underline{c}_1 = 0, \qquad \underline{c}_2 = 0. \end{aligned} \tag{3.25}$$

By solving the above equations and by condition (2.10) we have $\overline{c}_1 = \underline{c}_1 = a_1/b_1$ and $\overline{c}_2 = \underline{c}_2 = 0$. This proves $\overline{\mathbf{c}} = \underline{\mathbf{c}} = (a_1/b_1, 0)$. A similar argument as that in the proof of Theorem 2.2 shows that the property (3.7) holds for the above pair $\widehat{\mathbf{c}}$ and $\widetilde{\mathbf{c}}$. By Theorem 3.1(iii), the solution $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t))$ converges uniformly to $(a_1/b_1, 0)$ as $t \to \infty$. The conclusion in (iii) is proved. The proof of the conclusion in (iv) is similar. □

References

[1] Z. Liu, R. Tan, Y. Chen, L. Chen, On the stable periodic solutions of a delayed two-species model of facultative mutualism, Appl. Math. Comput. 196 (2008) 105–117.
[2] S. Lu, On the existence of positive periodic solutions to a Lotka–Volterra cooperative population model with multiple delays, Nonlinear Anal. 68 (2008) 1746–1753.
[3] Y. Saito, The necessary and sufficient condition for global stability of a Lotka–Volterra cooperative or competition system with delays, J. Math. Anal. Appl. 268 (2002) 109–124.
[4] F. Wei, K. Wang, Asymptotically periodic solution of N-species cooperation system with time delay, Nonlinear Anal. RWA 7 (2006) 591–596.
[5] H. Wu, Y. Xia, M. Lin, Existence of positive periodic solution of mutualism system with several delays, Chaos Solitons Fractals 36 (2008) 487–493.
[6] M. Delgado, A. Suárez, Stability and uniqueness for cooperative degenerate Lotka-Volterra model, Nonlinear Anal. 49 (2002) 757–778.
[7] P. Feng, Global and blow-up solutions for a mutualistic model, Nonlinear Anal. 68 (2008) 1898–1908.
[8] J. Huang, X. Zou, Traveling wavefronts in diffusive and cooperative Lotka-Volterra system with delays, J. Math. Anal. Appl. 271 (2002) 455–466.
[9] K.I. Kim, Z. Lin, Blowup in a three-species cooperating model, Appl. Math. Lett. 17 (2004) 89–94.
[10] K.I. Kim, Z. Lin, Blowup estimates for a parabolic system in a three-species cooperating model, J. Math. Anal. Appl. 293 (2004) 663–676.
[11] H. Li, M. Wang, Critical exponents and lower bounds of blow-up rate for a reaction-diffusion system, Nonlinear Anal. 63 (2005) 1083–1093.
[12] Y. Lou, Necessary and sufficient condition for the existence of positive solutions of certain cooperative system, Nonlinear Anal. 26 (1996) 1079–1095.
[13] Y. Lou, T. Nagylaki, W.-M. Ni, On diffusion-induced blowups in a mutualistic model, Nonlinear Anal. 45 (2001) 329–342.
[14] R.S. Cantrell, C. Cosner, V. Hutson, Permanence in ecological system with spatial heterogeneity, Proc. Roy. Soc. Edinburgh 123A (1993) 533–559.
[15] P. Korman, A. Leung, On the existence and uniqueness of positive steady state in the Volterra–Lotka ecological models with diffusion, Appl. Anal. 26 (1987) 145–160.
[16] Z. Li, P.D. Mottoni, Bifurcation for some systems of cooperative and predator-prey type, J. Partial Differential Equations 5 (1992) 25–36.
[17] Z. Lu, Y. Takeuchi, Permanence and global stability for cooperative Lotka-Volterra diffusion systems, Nonlinear Anal. 19 (1992) 963–975.
[18] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.
[19] Y.-M. Wang, Asymptotic behavior of solutions for a cooperation-diffusion model with a saturating interaction, Comput. Math. Appl. 52 (2006) 339–350.
[20] L. Wang, K. Li, On positive solutions of the Lotka-Volterra cooperating models with diffusion, Nonlinear Anal. 53 (2003) 1115–1125.
[21] W. Feng, X. Lu, Some coexistence and extinction results in a three species ecological model, Differential Integral Equations 8 (1995) 617–626.
[22] C.V. Pao, Quasisolutions and global attractor of reaction-diffusion systems, Nonlinear Anal. 48 (2002) 349–362.
[23] Y. Yamada, Stability of steady-states for prey–predator diffusion equations with homogeneous Dirichlet conditions, SIAM J. Math. Anal. 21 (1990) 327–345.
[24] C.V. Pao, Convergence of solutions of reaction-diffusion systems with time delays, Nonlinear Anal. 48 (2002) 349–362.
[25] C.V. Pao, Global asymptotic stability of Lotka-Volterra 3-species reaction-diffusion systems with time delays, J. Math. Anal. Appl. 281 (2003) 186–204.
[26] C.V. Pao, Global asymptotic stability of Lotka-Volterra competition systems with diffusion and time delays, Nonlinear Anal. RWA 5 (2004) 91–104.
[27] C.V. Pao, The Global attractor of a competitor–competitor-mutualist reaction-diffusion system with time delays, Nonlinear Anal. 48 (2007) 2623–2631.
[28] Y.-M. Wang, Global asymptotic stability of 3-species Lotka-Volterra models with diffusion and time delays, Appl. Math. Comput. 195 (2008) 34–48.
[29] R.P. Agarwal, Difference Equations and Inequalities, Marcel Dekker Inc, New York, 1992.
[30] G.S. Ladde, V. Lakshmikantham, V.A. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston, 1985.
[31] P.Y.H. Pang, R.P. Agarwal, Monotone iterative methods for a general class of discrete boundary value problems, Comput. Math. Appl. 28 (1994) 243–254.
[32] Q. Sheng, R.P. Agarwal, Monotone methods for higher-order partial difference equations, Comput. Math. Appl. 28 (1994) 291–307.
[33] A. Berman, R. Plemmons, Nonnegative Matrix in the Mathematical Science, Academic Press, New York, 1979.
[34] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.